Int J Solids Structures Vol 22, No 3, pp. 257–266, 1986
Printed in Great Britain

0020–7683/86 $3 00 + 00

COMPARISON OF BUCKLING DEFORMATIONS IN COMPRESSED ELASTIC COSSERAT PLATES WITH THREE-DIMENSIONAL PLATES

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(*Received* 15 *May* 1984)

Abstract—A rectangular elastic Cosserat plate under uniform uniaxial compression can buckle in a bending mode at low compression levels, or in a bulging mode at higher levels. Shear deformation in each mode is governed by a corresponding constitutive coefficient, denoted by α_3 in the bending mode and by α_8 in the bulging mode. By comparing the eigenconditions in the bending mode for a Cosserat plate and a three-dimensional plate, a value for the shear coefficient α_3 is determined. However, a comparison of eigenconditions for the bulging mode discloses an anomaly, with the result that a value for α_8 cannot be obtained by this method.

1. INTRODUCTION

Biot[1, 2]† was apparently the first to treat plate buckling using three-dimensional elasticity theory, and obtained an eigencondition in the bending mode for a linear elastic isotropic plate of arbitrary thickness. Subsequent work has mostly concerned plates of incompressible or nearly incompressible materials. A comprehensive treatment of buckling of thick elastic plates under compression, which includes a survey of work on the problem, is presented in recent papers by Sawyers and Rivlin[3, 4]. Buckling of compressed elastic Cosserat plates has been treated by Green and Naghdi[5] as an application of the method of superposing small deformations on a large deformation of a Cosserat surface. Since the coefficients in a quadratic strain energy function for an elastic Cosserat plate are given explicitly by Naghdi in [6], it is convenient to compare results for a Cosserat plate with those of a three-dimensional plate which also possesses a quadratic strain energy function. Accordingly, the present paper includes results for a three-dimensional plate with a quadratic strain energy function, obtained using the three-dimensional theory of small deformations superposed on a large deformation[7]. While the strains associated with the bulging mode in both a Cosserat plate and a three-dimensional plate are so large that they could not exist in materials for which a quadratic strain energy function is appropriate, these results are included since they are of some theoretical interest in comparing a Cosserat plate with a three-dimensional plate. These results could be useful in connection with buckling of elastic–plastic plates.

2. UNIFORM FINITE EXTENSION OF A COSSERAT PLATE UNDER UNIAXIAL COMPRESSION

Green and Naghdi[5] treat flexural buckling of a square Cosserat plate under equal biaxial compression. In the following, their treatment is modified for the case of uniaxial compression. The notation and terminology of [5] are used, except for the constitutive coefficients which are stated in the notation of [6]. The initial large deformation is a state of uniform finite extension, characterized now by the distinct extension ratios $k_1 \leqslant 1$ in the direction of compression, and $k_2 \geqslant 1$ in the perpendicular direction. Also, there is uniform thickness change described by the director component $d_3 = d \geqslant 1$. In the state of uniform finite extension, convected surface coordinates θ^α coincide with fixed rectangular Cartesian coordinates x_1, x_2 lying in the middle surface. The covariant and contravariant components of the surface metric tensor referred to θ^α coordinates

† The result in [1] contains a misprint; the corrected form is given in [2].

in the undeformed plate are denoted by $A_{\alpha\beta}$ and $A^{\alpha\beta}$, and in the currently deformed plate by $a_{\alpha\beta}$ and $a^{\alpha\beta}$. Hence,

$$A_{\alpha\beta} = \begin{bmatrix} k_1^{-2} & 0 \\ 0 & k_2^{-2} \end{bmatrix}, \quad A^{\alpha\beta} = \begin{bmatrix} k_1^2 & 0 \\ 0 & k_2^2 \end{bmatrix}, \quad a_{\alpha\beta} = a^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.1}$$

The strain measures describing the uniform finite extension are given by

$$e_{11} = (1 - k_1^{-2})/2, \qquad e_{22} = (1 - k_2^{-2})/2, \qquad \delta_3 = d - 1, \tag{2.2}$$

with

$$e_{12} = 0, \qquad \delta_\alpha = d_\alpha = 0, \qquad \kappa_{i\alpha} = 0. \tag{2.3}$$

The externally applied compressive load acts parallel to the x_1 axis. The components $N^{\alpha i}$ of the contact force, the components $M^{\alpha i}$ of the contact couple, and the components π^i of the intrinsic surface director force referred to θ^α coordinates are given by

$$N^{\alpha\beta} = \begin{bmatrix} N^{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad N^{\alpha 3} = 0, \qquad M^{\alpha i} = 0, \qquad \pi^i = 0, \tag{2.4}$$

where N^{11} is a constant. The equilibrium conditions are satisfied trivially. These force and couple components, except for the transverse shear components $N^{\alpha 3}$, are related to the kinematic variables $e_{\alpha\beta}$, $\kappa_{i\alpha}$ and δ_i by constitutive relations. For arbitrary deformation of a homogeneous isotropic plate with a quadratic strain energy function, these relations are [6],

$$(a/A)^{1/2}\overline{N}^{\alpha\beta} = (\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + 2\alpha_2 A^{\alpha\gamma}A^{\beta\delta})e_{\gamma\delta} + \alpha_9 A^{\alpha\beta}\delta_3, \tag{2.5}$$

$$\overline{N}^{\alpha\beta} = N^{\alpha\beta} - \pi^\alpha d^\beta - M^{\gamma\alpha}\lambda^{\beta}_{\cdot\gamma}, \tag{2.6}$$

and λ^β_γ is a measure of bending deformation defined by [5], eqn (2.21), which is zero for uniform finite extension of a homogeneous plate. Also,

$$(a/A)^{1/2}\pi^\alpha = \alpha_3 A^{\alpha\gamma}\delta_\gamma, \tag{2.7}$$

$$(a/A)^{1/2}\pi^3 = \alpha_4\delta_3 + \alpha_9 A^{\alpha\beta}e_{\alpha\beta}, \tag{2.8}$$

$$(a/A)^{1/2}M^{\beta\alpha} = (\alpha_5 A^{\alpha\beta}A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma}A^{\beta\delta} + \alpha_7 A^{\alpha\delta}A^{\beta\gamma})\kappa_{\gamma\delta}, \tag{2.9}$$

$$(a/A)^{1/2}M^{\alpha 3} = \alpha_8 A^{\alpha\gamma}\kappa_{3\gamma}. \tag{2.10}$$

where

In eqns (2.5)–(2.10), a, A are the determinants of $a_{\alpha\beta}$, $A_{\alpha\beta}$, and the constitutive coefficients $\alpha_1, \alpha_2, \ldots, \alpha_9$ are given by

$$\begin{aligned} &\alpha_1 = \alpha_9 = \frac{\nu(1-\nu)}{1-2\nu}C, \qquad \alpha_2 = \frac{1-\nu}{2}C, \qquad \alpha_4 = \frac{(1-\nu)^2}{1-2\nu}C, \\ &\alpha_5 = \nu Ch^2/12, \qquad \alpha_6 = \alpha_7 = (1-\nu)Ch^2/24, \\ &\alpha_3 = 5(1-\nu)C/12, \qquad \alpha_8 = 7(1-\nu)Ch^2/240, \end{aligned} \tag{2.11}$$

and

$$C = Eh/(1 - \nu^2),$$

where E is Young's modulus, ν Poisson's ratio, and h the uniform thickness of the undeformed plate.

The extension ratio k_2 and director displacement δ_3 can be expressed in terms of k_1 by using the conditions $N^{22} = \overline{N}^{22} = 0$ and $\pi^3 = 0$. Thus

$$k_2^2 = 1 + \nu(1 - k_1^2), \qquad \delta_3 = \nu(1 - k_1^2)/2. \tag{2.12}$$

Also, $N^{11} = \overline{N}^{11}$, and finally N^{11}, can be expressed in terms of k_1 when eqns (2.12) are used. It is convenient to introduce N_0^{11} as a measure of the axial load by putting

$$N_0^{11} = (a/A)^{1/2}N^{11} = -C(1 - \nu^2)k_1^2(1 - k_1^2)/2. \tag{2.13}$$

Relation (2.13) illustrates a characteristic peculiarity of a quadratic strain energy function, namely, that axial compression rises to a maximum value as k_1 decreases from unity, and then decreases to zero as $k_1 \to 0$. A three-dimensional plate behaves similarly.

3. SUPERPOSED SMALL SINUSOIDAL BUCKLING DEFORMATIONS ON UNIAXIAL COMPRESSION IN A COSSERAT PLATE

Small buckling deformations are now considered to occur in a Cosserat plate loaded in compression, which correspond to superposed small plane-strain deformations in a three-dimensional plate. Buckling deformations in a Cosserat plate are described by an incremental displacement vector and an incremental director displacement vector with components u_i and b_i, respectively, referred to θ^α coordinates in the state of uniform finite extension. In this configuration, it is recalled that the convected coordinates θ^α coincide with rectangular Cartesian coordinates. The buckling deformations considered here are characterized by putting

$$u_i = (u_1, 0, w), \qquad b_i = (b_1, 0, b_3), \tag{3.1}$$

where u_1, w, b_1 and b_3 are functions of θ^1 only.

The increments to the kinematic variables $e_{\alpha\beta}$, $\kappa_{i\alpha}$ and δ_i are denoted by $e'_{\alpha\beta}$, $\kappa'_{i\alpha}$ and δ'_i. In view of eqns (2.1) and (3.1), they are

$$e'_{11} = u_{1,1}, \qquad e'_{12} = e'_{22} = 0, \tag{3.2}$$

$$\kappa'_{11} = b_{1,1}, \qquad \kappa'_{31} = b_{3,1}, \qquad \kappa'_{12} = \kappa'_{21} = \kappa'_{22} = \kappa'_{32} = 0, \tag{3.3}$$

$$\delta'_1 = b_1 + dw_{,1}, \qquad \delta'_3 = b_3, \qquad \delta'_2 = 0. \tag{3.4}$$

The components of the force and couple resultants referred to convected coordinates θ^α in the buckled plate are denoted by $N^{\alpha i} + \epsilon N'^{\alpha i}$, $M^{\alpha i} + \epsilon M'^{\alpha i}$ and $\pi^i + \epsilon\pi'^i$, where $\epsilon = 1$ is a coefficient which identifies quantities which vanish with the superposed buckling deformation. The constitutive relations for the nonzero components of $N'^{\alpha\beta}$, $M'^{\alpha i}$ and π'^i, obtained from eqns (2.5)–(2.10) using (3.2)–(3.4), are

$$\overline{N}'^{11} = N'^{11} = [-N^{11} + kk_1^4(\alpha_1 + 2\alpha_2)]u_{1,1} + kk_1^2\alpha_9 b_3, \tag{3.5}$$

$$\overline{N}'^{22} = N'^{22} = k[\alpha_1 k_1^2 k_2^2 u_{1,1} + \alpha_9 k_2^2 b_3], \tag{3.6}$$

$$M'^{11} = kk_1^4(\alpha_5 + \alpha_6 + \alpha_7)b_{1,1}, \tag{3.7}$$

$$M'^{22} = kk_1^2 k_2^2 \alpha_5 b_{1,1}, \tag{3.8}$$

$$M'^{13} = kk_1^2\alpha_8 b_{3,1} \tag{3.9}$$

$$\pi'^1 = kk_1^2\alpha_3(b_1 + dw_{,1}), \tag{3.10}$$

$$\pi'^3 = k(\alpha_4 b_3 + k_1^2\alpha_9 u_{1,1}), \tag{3.11}$$

where $k = (A/a)^{1/2} = (k_1k_2)^{-1}$.

There are five nontrivial equilibrium conditions for determining the four displacement components u_1, w, b_1 and b_3, and the transverse shear N'^{13}. They are

$$2N^{11}u_{1,11} + N'^{11}{}_{,1} = 0, \qquad N'^{13}{}_{,1} + N^{11}w_{,11} = 0, \tag{3.12}$$

$$M'^{11}{}_{,1} = \pi'^1, \qquad M'^{13}{}_{,1} = \pi'^3, \qquad N'^{13} - \pi'^1 d = 0. \tag{3.13}$$

The transverse shear N'^{13} can be eliminated between $(3.12)_2$ and $(3.13)_3$. The result is

$$d\pi'^1{}_{,1} + N^{11}w_{,11} = 0. \tag{3.14}$$

Equations $(3.12)_1$, $(3.13)_1$, $(3.13)_2$ and (3.14) now comprise a system of four equations for u_1, w, b_1 and b_3, when the constitutive relations (3.5)–(3.11) are used. As noted in [5], these equations uncouple into two sets, one for w and b_1, which describe bending deformation, and the other for u_1 and b_3, which describe symmetric bulging or thinning of the plate about the middle surface.

Buckling in the bending mode is governed by the two equations, which follow from eqns (3.7), (3.10), $(3.13)_1$ and (3.14),

$$k_1^2 d\alpha_3(b_{1,1} + dw_{,11}) + N_0^{11}w_{,11} = 0, \tag{3.15}$$

$$k_1^2(Ch^2/12)b_{1,11} - \alpha_3(b_1 + dw_{,1}) = 0. \tag{3.16}$$

In obtaining eqns (3.15) and (3.16), use is made of the expressions (2.11) for the constitutive coefficients. However, α_3 in (3.15) and (3.16) is left unspecified. A solution of (3.15) and (3.16) for sinusoidal buckling can be taken as

$$w = A\sin(\pi\theta^1/l), \qquad b_1 = B\cos(\pi\theta^1/l), \tag{3.17}$$

where A and B are arbitrary constants, and l is the half wavelength. For $A, B \neq 0$, eqns (3.15) and (3.16) are satisfied, provided

$$-(k_1^4d^2)^{-1}N_0^{11} = \frac{\alpha_3 C\pi^2h^2/(12l^2)}{\alpha_3 + k_1^2C\pi^2h^2/(12l^2)}. \tag{3.18}$$

The eigencondition (3.18) is now simplified by putting $k_1 = 1 - P/E$, where $P/E \ll 1$. P can be identified with uniform compressive stress in a three-dimensional plate. When just first-order terms in P/E and h^2/l^2 are retained, the eigencondition becomes, in view of eqns (2.2), $(2.12)_2$ and (2.13),

$$(1 - \nu^2)\frac{P}{E} = \frac{\pi^2h^2}{12l^2}. \tag{3.19}$$

Thus the familiar Euler formula is recovered.

The equations governing the bulging mode, obtained from the equilibrium conditions $(3.12)_1$ and $(3.13)_2$, along with the constitutive relations (3.5), (3.9) and (3.11), are

$$[N_0^{11} + k_1^4(\alpha_1 + 2\alpha_2)]u_{1,11} + k_1^2\alpha_9 b_{3,1} = 0, \tag{3.20}$$

$$k_1^2\alpha_8 b_{3,11} - \alpha_4 b_3 - k_1^2\alpha_9 u_{1,1} = 0. \tag{3.21}$$

A solution for sinusoidal buckling can be constructed by putting

$$u_1 = A\sin(\pi\theta^1/l), \qquad b_3 = B\cos(\pi\theta^1/l), \tag{3.22}$$

where A and B are arbitrary constants. The eigencondition for a nontrivial solution can be written

$$\frac{\alpha\pi^2h^2}{l^2} = \frac{1}{k_1^2}\cdot\frac{(1-\nu)^2(1-k_1^2)(2k_1^2)^{-1}-1}{1-(1-2\nu)(1-\nu)^{-2}(1-\nu^2)(1-k_1^2)(2k_1^2)^{-1}}, \tag{3.23}$$

where the coefficient α has been introduced, such that

$$\alpha_8 = \alpha C h^2. \tag{3.24}$$

Solutions exist just for values of k_1 for which the right side of (3.23) is positive, that is, for

$$\frac{1-\nu^2}{3-\nu^2} > k_1^2 > \frac{(1-2\nu)(1+\nu)}{3-3\nu-2\nu^2}. \tag{3.25}$$

As a typical example, for $\nu = 0.25$, k_1 must satisfy the bounds $0.565 > k_1 > 0.542$, and in this rather narrow range of values for k_1, the thickness/wavelength ratio varies between zero and infinity, $0 < h/l < \infty$. The value of h/l increases as k_1 decreases, that is, as deformation increases. Thus the eigencondition (3.23) predicts that a thin plate is more susceptible to buckling in the bulging mode than a thick plate. Compared to a three-dimensional plate, this behavior is anomalous.

4. UNIFORM FINITE EXTENSION OF A THREE-DIMENSIONAL PLATE UNDER UNIAXIAL COMPRESSION

In a three-dimensional rectangular plate in a state of uniform finite extension, convected coordinates θ^i are introduced which coincide with fixed rectangular Cartesian coordinates y_i referred to axes parallel to the edges of the plate. The y_1 and y_3 axes lie in the middle surface of the plate, with the y_1 axis in the direction of externally applied uniform compressive load. The notation of [7] is followed, in which the covariant and contravariant metric tensors referred to the convected coordinates θ^i are denoted by g_{ij} and g^{ij} in the undeformed plate, and by G_{ij} and G^{ij} in the plate deformed by uniform finite extensions λ_i. Hence, from [7], eqns (4.23) and (4.24),

$$g_{ij} = \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \lambda_1^{2} & 0 & 0 \\ 0 & \lambda_2^{2} & 0 \\ 0 & 0 & \lambda_3^{2} \end{bmatrix}, g = (\lambda_1^2\lambda_2^2\lambda_3^2)^{-1}, \tag{4.1}$$

and

$$G_{ij} = G^{ij} = \delta_{ij}, \qquad G = 1, \tag{4.2}$$

where $g = |g_{ij}|$, $G = |G_{ij}|$ and δ_{ij} is the Kronecker delta. The contravariant symmetric stress tensor τ^{ij} referred to the coordinates θ^i in the state of uniform finite extension has the form

$$\tau^{ij} = \begin{bmatrix} -P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.3}$$

where P is a constant which represents uniform compressive stress per unit area in the currently deformed cross-section θ^1 = constant. For an isotropic plate under loading described by eqn (4.3),

$$\lambda_2 = \lambda_3. \tag{4.4}$$

Constitutive relations for the stress components τ^{ij} are derived from a strain energy function. For an isotropic material, a strain energy function which is quadratic in the strain components $\gamma_{ij} = (G_{ij} - g_{ij})/2$ can be formed from the strain invariants I_1 and I_2 defined by [7], eqn (2.2.23), as

$$I_1 = 3 + 2g^{ij}\gamma_{ij}, \tag{4.5}$$

$$I_2 = 3 + 4g^{ij}\gamma_{ij} + 2g^{ij}g^{mn}(\gamma_{ij}\gamma_{mn} - \gamma_{im}\gamma_{jn}). \tag{4.6}$$

For uniform finite extension, these invariants can be written

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2. \tag{4.7}$$

The strain energy function

$$W = \frac{\lambda + 2\mu}{8}(I_1 - 3)^2 + \frac{\mu}{2}(2I_1 - I_2 - 3), \tag{4.8}$$

where λ and μ are the Lamé constants, coincides with the usual quadratic strain energy function of linear isotropic elasticity. For finite deformation, W represents strain energy per unit volume in the undeformed body. The constitutive relations[7], eqn (4.2.7), for uniform finite extension have the typical form

$$\tau^{11} = \Phi\lambda_1^2 + \Psi\lambda_1^2(\lambda_2^2 + \lambda_3^2) + p, \tag{4.9}$$

where

$$\Phi = 2(g/G)^{1/2}\frac{\partial W}{\partial I_1} = 2(\lambda_1\lambda_2\lambda_3)^{-1}[(\lambda + 2\mu)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)/4 + \mu], \tag{4.10}$$

$$\Psi = 2(g/G)^{1/2}\frac{\partial W}{\partial I_2} = -\mu(\lambda_1\lambda_2\lambda_3), \tag{4.11}$$

$$p = 2(G/g)^{1/2}\frac{\partial W}{\partial I_3} = 0, \qquad I_3 = G/g, \tag{4.12}$$

when evaluated using the strain energy function (4.8). An equation for τ^{22} can be obtained from eqn (4.9) by the cyclic change of indices $1 \to 2 \to 3 \to 1$. In view of eqns (4.3) and (4.4), the constitutive equations for τ^{11} and τ^{22} yield the relations

$$P = \mu\lambda_1(1 - \lambda_1^2/\lambda_2^2), \tag{4.13}$$

$$\lambda_2^2 = (\lambda + \mu)^{-1}[\lambda(3 - \lambda_1^2)/2 + \mu]. \tag{4.14}$$

From eqns (4.13) and (4.14) it can be noted that P at first increases as λ_1 decreases from unity, and then decreases to zero as $\lambda_1 \to 0$. Thus a three-dimensional plate with a quadratic strain energy function exhibits similar behavior to a Cosserat plate as noted in eqn (2.13). The extension ratios λ_1, $\lambda_2 = \lambda_3$ in the three-dimensional plate can be identified with the extension ratios k_1 and k_2 in the Cosserat plate. However, a difference is to be noted in that the director component $d_3 = d$, which describes change of thickness in the Cosserat plate, does not coincide with the extension ratio λ_2 in the three-dimensional plate.

5. SUPERPOSED SMALL SINUSOIDAL BUCKLING DEFORMATIONS ON UNIAXIAL COMPRESSION IN A THREE-DIMENSIONAL PLATE

A plane-strain perturbation describing buckling is now superposed on the uniform finite extension of the plate. The incremental displacements are described by a vector with components w_i referred to the convected coordinates θ^i in the configuration (4.2), with

$$w_1 = w_1(\theta^1, \theta^2), \qquad w_2 = w_2(\theta^1, \theta^2), \qquad w_3 = 0. \tag{5.1}$$

Referred to the convected coordinates θ^i in the buckled plate, the components of the contravariant symmetric stress tensor are denoted by $\tau^{ij} + \epsilon\tau'^{ij}$, where $\epsilon = 1$ is again a coefficient which identifies small quantities that vanish with the perturbation. Constitutive equations for the stress increments τ'^{ij} are given directly in terms of w_i by [7], eqns (4.2.12) and (4.2.13), and have the form

$$\tau'^{11} = c_{11}w_{1,1} + c_{12}w_{2,2}, \tag{5.2}$$

$$\tau'^{22} = c_{21}w_{1,1} + c_{22}w_{2,2}, \tag{5.3}$$

$$\tau'^{12} = c_{66}(w_{1,2} + w_{2,1}), \tag{5.4}$$

when account is taken of eqn (5.1). Also,

$$\tau'^{33} = \tau'^{33}(\theta^1, \theta^2), \qquad \tau'^{13} = \tau'^{23} = 0. \tag{5.5}$$

For the strain energy function (4.8), the constitutive coefficients appearing in eqns (5.2)–(5.4) reduce to

$$c_{11} = P + (\lambda + 2\mu)\lambda_1^3\lambda_2^{-2}, \tag{5.6}$$

$$c_{22} = (\lambda + 2\mu)\lambda_2^2\lambda_1^{-1}, \tag{5.7}$$

$$c_{12} = -2\lambda_1\lambda_2^{-2}[(\lambda + 2\mu)(\lambda_1^2 + 2\lambda_2^2 - 3)/4 + \mu] + (\lambda + 2\mu)\lambda_1, \tag{5.8}$$

$$c_{21} = -P + c_{12}, \tag{5.9}$$

$$c_{66} = \mu\lambda_1, \tag{5.10}$$

when (4.3) and (4.4) are used. Relations (5.2)–(5.10) coincide with the usual relations of linear isotropic elasticity for plane strain when $P = 0$ and $\lambda_1 = \lambda_2 = 1$. In view of eqns (5.1) and (5.2)–(5.5), the equilibrium conditions for the perturbation[7], eqn (4.2.21),

$$(\tau'^{ij} + \tau^{ir}w_{j,r} + \tau^{rj}w_{i,r})_{,i} = 0, \tag{5.11}$$

reduce to

$$\tau'^{11}{}_{,1} + \tau'^{12}{}_{,2} - 2Pw_{1,11} - Pw_{2,12} = 0, \tag{5.12}$$

$$\tau'^{12}{}_{,1} + \tau'^{22}{}_{,2} - Pw_{2,11} = 0, \tag{5.13}$$

since $\tau^{11} = -P$ is the only nonzero component of prestress in eqn (4.3). When the constitutive relations (5.2)–(5.4) are used, the equilibrium equations (5.12) and (5.13) become

$$(c_{11} - 2P)w_{1,11} + (c_{21} + c_{66})w_{2,12} + c_{66}w_{1,22} = 0, \tag{5.14}$$

$$(c_{66} - P)w_{2,11} + (c_{21} + c_{66})w_{1,12} + c_{22}w_{2,22} = 0. \tag{5.15}$$

Solutions of eqns (5.14) and (5.15) for sinusoidal buckling can be taken as

$$w_1 = -\sin(\pi\theta^1/l)[C_1\mu_1 \sinh(\pi\rho_1\theta^2/l) + C_2\mu_2 \sinh(\pi\rho_2\theta^2/l)], \tag{5.16}$$

$$w_2 = \cos(\pi\theta^1/l)[C_1 \cosh(\pi\rho_1\theta^2/l) + C_2 \cosh(\pi\rho_2\theta^2/l)], \tag{5.17}$$

or

$$w_1 = -\sin(\pi\theta^1/l)[C_3\mu_1 \cosh(\pi\rho_1\theta^2/l) + C_4\mu_2 \cosh(\pi\rho_2\theta^2/l)], \tag{5.18}$$

$$w_2 = \cos(\pi\theta^1/l)[C_3 \sinh(\pi\rho_1\theta^2/l) + C_4 \sinh(\pi\rho_2\theta^2/l)], \tag{5.19}$$

where $\pm\rho_1$ and $\pm\rho_2$ are the roots of the fourth-order equation in ρ,

$$c_{22}c_{66}\rho^4 - [c_{66}(c_{66} - P) + c_{22}(c_{11} - 2P) - (c_{21} + c_{66})^2]\rho^2 + (c_{66} - P)(c_{11} - 2P) = 0, \tag{5.20}$$

and

$$\mu_1 = \frac{(c_{21} + c_{66})\rho_1}{c_{11} - 2P - c_{66}\rho_1^2}, \qquad \mu_2 = \frac{(c_{21} + c_{66})\rho_2}{c_{11} - 2P - c_{66}\rho_2^2},$$

and $C_1, \ldots, C_4$ are arbitrary constants. Expressions (5.16) and (5.17) describe bending of the plate, while (5.18) and (5.19) describe symmetric bulging or thinning of the plate about the middle surface.

The surfaces of the plate $\theta^2 = \pm\lambda_2 h/2$, where h is the thickness of the undeformed plate, are free from applied tractions. Hence the boundary condition[7], eqn (4.2.30), assumes the form

$$\tau'^{12} = \tau'^{22} = 0, \qquad \theta^2 = \pm\lambda_2 h/2. \tag{5.21}$$

The eigencondition for the bending mode is obtained by requiring that the expressions (5.16) and (5.17) satisfy the boundary conditions (5.21) when $C_1, C_2 \neq 0$. The eigencondition for the bending mode is

$$\frac{\rho_2}{\rho_1}\cdot\frac{\mu_1\rho_1 + 1}{\mu_2\rho_2 + 1}\cdot\frac{c_{22}\rho_2 - c_{21}\mu_2}{c_{22}\rho_1 - c_{21}\mu_1} = \frac{\rho_2}{\rho_1}\cdot\frac{\tanh[(\pi\rho_1\lambda_2 h)/(2l)]}{\tanh[(\pi\rho_2\lambda_2 h)/(2l)]}. \tag{5.22}$$

Similarly, the eigencondition for the bulging mode is

$$\frac{\rho_1}{\rho_2}\cdot\frac{\mu_1\rho_1 + 1}{\mu_2\rho_2 + 1}\cdot\frac{c_{22}\rho_2 - c_{11}\mu_2}{c_{22}\rho_1 - c_{21}\mu_1} = \frac{\rho_1}{\rho_2}\cdot\frac{\tanh[(\pi\rho_2\lambda_2 h)/(2l)]}{\tanh[(\pi\rho_1\lambda_2 h)/(2l)]}. \tag{5.23}$$

The eigencondition for the bending mode (5.22) can be expanded as a double series in P/E, h/l when $P/E \ll 1$ and $h/l \ll 1$. To proceed, it is noted first that when $P = 0$ and $\lambda_1 = \lambda_2 = 1$, the roots of (5.20) are $\rho_1^2 = \rho_2^2 = 1$. Thus the leading term on each side of (5.22) is unity. The left side of (5.22) is expanded to the second order in the small quantities $(\rho_1^2 - 1)$, $(\rho_2^2 - 1)$. The right side is expanded in a series in h/l, which is even in h/l. The leading terms of unity cancel; $(\rho_1^2 - \rho_2^2)$ then appears as a common factor in the remaining terms on both sides, and cancels. Then substitutions for λ_1, λ_2 are made by putting

$$\lambda_1 = 1 - P/E, \qquad \lambda_2 = 1 + \nu P/E. \tag{5.24}$$

Retention of just first-order terms in P/E and h^2/l^2 yields the Euler formula (3.19).

As λ_1 decreases from unity, the roots of eqn (5.20) are at first complex, and ρ_1, ρ_2 can be taken as a complex conjugate pair with positive real parts. In the limit as $h/l \to \infty$, the two eigenconditions, eqn (5.22) for the bending mode and eqn (5.23) for the bulging mode, come into coincidence and have the solutions

$$\lambda_1 = 0.625, 0.620, 0.612, \tag{5.25}$$

corresponding, respectively, to

$$\nu = 0.1, 0.25, 0.4.$$

The solutions describe localized surface instability.

For $h/l \ll 1$, solutions of the eigencondition (5.23) for the bulging mode can be represented approximately by

$$\beta(\lambda_1 - \lambda_1^*) = (\rho_2^2 - \rho_1^2)\left[\frac{\pi^2\lambda_2^2h^2}{12l^2} - (6\rho_2^2 + \rho_1^2)\frac{\pi^4\lambda_2^4h^4}{720l^4}\right], \tag{5.26}$$

where λ_1^* is the value of λ_1 for which the left side of eqn (5.23) equals unity, coincident roots of (5.20) excluded, and β is the coefficient of the first term in the Taylor series expansion of the left side about $\lambda_1 = \lambda_1^*$. Typical values of λ_1^* are

$$\lambda_1^* = 0.576, 0.565, 0.544, \tag{5.27}$$

and correspond, respectively, to

$$\nu = 0.1, 0.25, 0.4.$$

The roots of eqn (5.20) are now pure imaginary, that is, $\rho_1^2, \rho_2^2 < 0$, and when the roots are indexed such that $\rho_2^2 - \rho_1^2 > 0$, then $\beta > 0$. Relation (5.26) shows that a thick plate buckles at a larger value of λ_1 than a thin plate. The trend for a three-dimensional plate is opposite to that for a Cosserat plate governed by eqn (3.23). Also, it is inferred by eqn (5.26) that there are no solutions for $\lambda_1 < \lambda_1^*$.

Sawyers and Rivlin[4] obtained the results, for an incompressible neo-Hookean material, that the critical value of λ_1 in the bulging mode is reached first for the limiting case $h/l \to \infty$, and that λ_1 decreases monotonically as h/l decreases. Similar results were also obtained by Burgess and Levinson[8] for a slightly compressible rubberlike material which reduces to the neo-Hookean material in the incompressible case. An exhaustive investigation of solutions of the eigencondition (5.23) for the bulging mode has not been carried out, but, in view of the results in [4] and [8], it can be expected that solutions for λ_1 lie in the range bounded by the values (5.25) and (5.27), with λ_1 decreasing as h/l decreases. Rather perplexing is the observation that the lower bound on λ_1 for the three-dimensional plate (5.27) coincides with the upper bound on k_1 for the Cosserat plate (3.25), at least for the three values of Poisson's ratio $\nu = 0.1, 0.25$

and 0.4. Thus there is no intersection of the admissible regions of λ_1 and k_1 for the bulging mode, and matching of the eigenconditions (3.23) for a Cosserat plate and (5.23) or (5.26) for a three-dimensional plate to determine α_8 is thwarted.

6 MATCHING CONDITION FOR DETERMINATION OF α_3

The eigenconditions (3.18) for a Cosserat plate and (5.22) for a three-dimensional plate for buckling in the bending mode can be matched most conveniently to determine the shear coefficient α_3 in the limit as $h/l \to \infty$. The eigencondition (3.18) can be written, in the limit as $h/l \to \infty$,

$$\alpha_3/(Eh) = \tfrac{1}{4}(1 - k_1^2)\left[1 + \frac{\nu}{2}(1 - k_1^2)\right]^{-2}, \tag{6.1}$$

when eqns $(2.2)_3$, (2.11), (2.12) and (2.13) are used. The extension ratio k_1 is now assigned the values (5.25), which then yield

$$\alpha_3/(Eh) = 0.287,\ 0.265,\ 0.247 \tag{6.2}$$

corresponding respectively to

$$\nu = 0.1,\ 0.25,\ 0.4. \tag{6.3}$$

The value for α_3 listed in eqn (2.11) was obtained by matching the solution for twisting of a rectangular Cosserat plate to the St. Venant solution for torsion of a three-dimensional plate. When α_3 as given in eqn (2.11) is evaluated for the values of Poisson's ratio (6.3), the results are, respectively,

$$\alpha_3/(Eh) = 0.379,\ 0.333,\ 0.298. \tag{6.4}$$

The values (6.2) are about 20% smaller than (6.4), but show the same trend to decrease as Poisson's ratio increases. While it would be desirable for application of the Cosserat theory to plate and shell buckling problems to determine α_3 for $h/l \ll 1$, such a determination would require the rather formidable expansion of the eigencondition (5.22) for a three-dimensional plate to the second order in the small quantities P/E and h^2/l^2.

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